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Estimating the Loss Factors of Plates with Constrained Layer Damping Treatments

Peter J. Torvik*
Air Force Institute of Technology, Xenia, Ohio 45385
and
Brian D. Runyon†
U.S. Air Force Research Laboratory, Wright-Patterson Air Force Base, Ohio 45433

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A method is developed and validated for the determination of approximate values of loss factors and frequencies of rectangular plates with general boundary conditions, fully covered on one side with a constrained layer damping treatment. The method, referred to as the method of equivalent lengths, is an extension for use on plates of the commonly used practice, through eigenvalue replacement, of applying the Ross, Ungar, and Kerwin analysis to the prediction of loss factors for beams with other than simply supported ends. Application of the method requires only that the modulus, thickness, and density of the substrate, constraining layer, and shear layer be known, as well as either the eigenvalue or natural frequency for the plate of aspect ratio and boundary condition to which the constrained layer treatment is to be applied. Predictions of the method are compared with results obtained by finite element methods for plates with various boundary conditions, including those with zero, one, two, three, and four free edges.

Nomenclature

A, B, C	=	amplitudes of displacement
AR	=	aspect ratio, a/b
a	=	length of plate
b	=	width of plate
C, F, S	=	boundary condition: clamped, free, or simply supported
D	=	bending stiffness of bare plate
D_1	=	stiffness of parallel plate and constraining layer, uncoupled
d_V	=	defined dimensionless quantity
E_S, E_C	=	Young's modulus: substrate beam or plate, constraining layer
G^*, G	=	shear modulus: complex value, real part
g^*	=	dimensionless complex shear parameter
$\tilde{g}^*$	=	dimensioned complex shear parameter
K_n	=	dimensioned eigenvalue of plate
L, L_{EQ}	=	length of simply supported plate
n, m	=	mode indices for simply supported beam or plate
p_n, q_m	=	wave numbers for simply supported plate
S_i	=	axial stiffness of substrate or constraining layer
t_S, t_C	=	thickness: substrate beam or plate, constraining layer
u_C, v_C	=	in-plane displacements of constraining layer
u_S, v_S	=	in-plane displacements of substrate plate
w	=	transverse displacement of plate and constraining layer
x, y	=	coordinates
Y	=	dimensionless coupling coefficient
δu, δv	=	constraining layer displacements relative to substrate plate
η_{MAT}	=	loss factor of shear layer
η_{SYS}	=	loss factor of system
λ	=	dimensionless frequency
υ	=	Poisson's ratio of plate and constraining layer
ρ_S, ρ_C, ρ_V	=	densities of substrate, constraining layer, and shear layer
$(\rho t)_{\mathrm{EFF}}$	=	mass per unit area of plate, constraining layer, and shear layer
ω	=	complex frequency of system
ω_0	=	real frequency of uncoupled system
$\Re\{\}$	=	real part, operator
$\Im\{\}$	=	imaginary part, operator

Superscript

$\cdot$	=	time derivative, operator

I. Introduction

ALTHOUGH suitable determinations of the loss factors and frequencies of rectangular plates with constrained layer damping treatments (CLDT) can be obtained through the use of a finite element analysis (FEA), the need to reconfigure a finite element model for each candidate set of parameters often makes it impractical to use FEA in the initial design of a CLDT. In consequence, there exists a need for a means of quickly obtaining suitable estimates so that design parameters may be selected to identify a near optimal configuration, which can then be refined through a limited number of computations using FEA. It is the objective of this work to develop such an approximate method.

With some preliminary work having been done by Plass [1] on symmetric sandwich plates with thick viscoelastic cores, the comprehensive analysis applicable to constrained layer damping treatments for the control of structural vibration of beams was presented in 1959 by the team of Ross, Ungar, and Kerwin (RUK) through a series of three papers. It is the last [2] of these that spawned extensive application. The classical RUK analysis for the configuration shown in Fig. 1 is limited in applicability to beams with simply supported ends.

However, within the classical assumptions and limitations, namely, that the longitudinal inertia is negligible, shear deformations

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*Professor Emeritus, Aeronautics and Engineering Mechanics, 1866 Winchester Road. Fellow AIAA.

†Aerospace Engineer, Propulsion Directorate. Member AIAA.

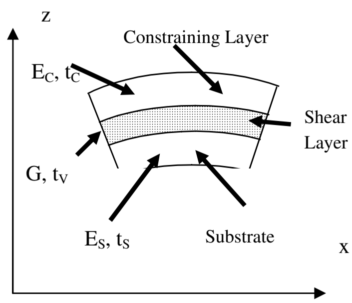

Fig. 1 The constrained layer damping treatment.

in the beam and constraining layer are negligible, the thickness of the shear layer remains constant, the strain energies due to extensions in the shear layer are negligible, the shear strain is uniform through the thickness of the shear layer, and all bonds are perfect; the system may be represented by a single differential equation of sixth order for the longitudinal deformation of the substrate or the constraining layer [3], or by a single sixth-order equation for the beam deflection [4]. Finally, Rao [5] applied Hamilton's principle to identify those boundary conditions that are consistent with the sixth-order differential equation for deflection. Rao [5] has also given analytical expressions, obtained from curve fits to computed data, which may be used to obtain loss factors for treated beams with general boundary conditions. However, these approximations are limited in applicability to values of loss factors for the damping layer that are much smaller than those in general practice.

As the application of the "exact" solution resulting from the Rao formulation for the beam is extremely difficult to apply in the case of the more general beam boundary conditions, an approximate technique has been accepted [6]. This is applied by replacing, in the RUK solution, the first eigenvalue of the simply supported beam with the eigenvalue for the beam mode of interest. This process may also be viewed as applying the RUK analysis to that simply supported beam of such length as to have the same frequency as that of the beam of length and boundary condition of interest. The loss factor predicted by the RUK theory for the treated simply supported beam of equivalent eigenvalue (or equivalent length) is then taken as an estimate of the loss factor for the desired configuration.

In the case of the plate with CLDT, the classical assumptions of the RUK theory may be applied and a sixth-order equation in two spatial variables developed. This was accomplished for the symmetric sandwich plate by Mead [7], and more generally, by Abdulhadi [8] and by Mead [9]. Rao and Nakra [10] have given the general boundary conditions.

Abdulhadi [8] hypothesized that, for a given CLDT configuration, the loss factor would depend only on the eigenvalue of the plate upon which it is mounted. However, he offered no justification or validation. Mead [9] noted the similarity between the solution for the treated plate with two opposing edges simply supported and the solution for the treated beam having the same boundary conditions as those of the other two edges of the plate. This suggests the possibility that a plate with a CLDT and *any* boundary condition might be related to an equivalent beam of some boundary condition. Then, by relating that equivalent beam to a simply supported beam of equivalent length, an approximate solution for the loss factor of a plate of any general boundary condition might be obtained from the RUK solution for a simply supported beam. Throughout this sequence, the properties, that is, the modulus, thickness, and density of each layer (substrate, constraining layer, and shear layer) would be retained.

This hypothesis will be explored in the remainder of this work. First, the equations of motion for a plate with CLDT will be given. It will then be shown that the solution for the loss factor of the simply supported plate is identical in form to that of the simply supported beam ensuring, in this case, the existence of an appropriate equivalent length. It is then hypothesized that the loss factor of a plate with CLDT and general boundary condition are the same as the loss factor of a simply supported plate with the same CLDT, but of such dimensions as to have the same eigenvalue. The length of the simply supported beam that is equivalent to that simply supported plate is then found, and the RUK solution is applied to that equivalent beam, using the original parameters for the CLDT. The resulting loss factor is then taken as that of the original plate. An estimate of the frequency is also obtained. This process, for brevity, will be termed the method of equivalent lengths (MEL).

The validity of this approach will be evaluated by comparing the results of such MEL predictions to values of system loss factor and frequency found by finite element methods. Rectangular plates with several boundary conditions and CLDT having thin constraining layers and thin shear layers of unit loss factor will be considered.

II. Plates with Constrained Layer Damping Treatments

The equations of motion for a plate of E_S and t_S, a constraining layer of E_C and t_C, and a shear layer of $G^* = G(1 + j\eta_{\mathrm{MAT}})$ and t_V may be found by invoking the customary assumptions of the RUK theory (see above). The strain energies included are only the following: 1) that due to bending about two axes; 2) that due to axial extension in the plate and constraining layer in two orthogonal in-plane directions; and 3) the strain energy due to shear in the shear layer. The total strain energy may be then written in terms of the plate deflection $w(x, y, t)$, two components of in-plane deformation in the plate, $u_S(x, y, t)$ and $v_S(x, y, t)$, and two components in the constraining layer, $u_C(x, y, t)$ and $v_C(x, y, t)$. Hamilton's principle may be applied and, if the kinetic energies due to rotations and in-plane displacements are also neglected, leads to a twelfth-order system of five differential equations: one of fourth order and four of second, together with appropriate boundary conditions. The details of this development are given in [10].

If it is then further assumed that the Poisson's ratio υ of the plate and the constraining layer are the same, the system may be reduced to an eighth-order system in three variables: the plate deflection, $w(x, y, t)$, and the in-plane deflections of the constraining layer *relative* to the plate, that is,

$$\delta u(x, y, t) = u_C(x, y, t) - u_S(x, y, t) \tag{1}$$

$$\delta v(x, y, t) = v_C(x, y, t) - v_S(x, y, t) \tag{2}$$

The resulting eighth-order system of three differential equations is

$$D_1\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - G\left[\frac{\partial \delta u}{\partial x} + d_V t_V \frac{\partial^2 w}{\partial x^2}\right] d_V - G\left[\frac{\partial \delta v}{\partial y} + d_V t_V \frac{\partial^2 w}{\partial y^2}\right] d_V + (\rho t)_T \left(\frac{\partial^2 w}{\partial t^2}\right) = 0 \tag{3}$$

$$-\frac{G}{t_V}\left(\frac{1}{S_C} + \frac{1}{S_S}\right)\left[\delta u + d_V t_V \frac{\partial w}{\partial x}\right] + \frac{\partial^2 \delta u}{\partial x^2} + \frac{(1-\upsilon)}{2}\frac{\partial^2 \delta u}{\partial y^2} + \frac{(1+\upsilon)}{2}\frac{\partial^2 \delta v}{\partial x \partial y}\Bigg) = 0 \tag{4}$$

$$-\frac{G_V}{t_V}\left(\frac{1}{S_C} + \frac{1}{S_S}\right)\left[\delta v + d_V t_V \frac{\partial w}{\partial y}\right] + \frac{\partial^2 \delta v}{\partial y^2} + \frac{(1+\upsilon)}{2}\frac{\partial^2 \delta u}{\partial y \partial x} + \frac{(1-\upsilon)}{2}\frac{\partial^2 \delta v}{\partial x^2} = 0 \tag{5}$$

where

$$D_1 = \sum \frac{E_i t_i^3}{12(1-\upsilon^2)} \tag{6}$$

$$S_i = \left\{\frac{E_i t_i}{1-\upsilon^2}\right\} \tag{7}$$

with "i" denoting either the substrate beam or constraining layer, and

$$d_V = 1 + (t_S + t_C)/2t_V \tag{8}$$

The total mass per unit area is $(\rho t)_T = \rho_S t_S + \rho_C t_C + \rho_V t_V$. These equations were obtained by taking the shear modulus to be a real quantity. The viscoelastic correspondence principle [11] is then invoked and the real valued G replaced by the complex modulus $G^* = G(1 + j\eta_{\mathrm{MAT}})$.

After introducing certain combinations of system parameters,

$$\tilde{g}^* = \frac{G^*}{t_V}\left(\frac{1}{S_C} + \frac{1}{S_S}\right) \tag{9}$$

$$Y = \frac{(t_V d_V)^2}{D_1}\left(\frac{1}{S_C} + \frac{1}{S_S}\right)^{-1} \tag{10}$$

the equation for plate bending , Eq. (3), may be rewritten as

$$\nabla^4 w - \tilde{g}^* \frac{Y}{d_V t_V}\left[\frac{\partial \delta u}{\partial x} + \frac{\partial \delta v}{\partial y}\right] - \tilde{g}^* Y \nabla^2 w + \frac{(\rho t)_T}{D_1}\left(\frac{\partial^2 w}{\partial t^2}\right) = 0 \tag{11}$$

After differentiating Eq. (4) with respect to x and Eq. (5) with respect to y and adding the two equations, we find that

$$-\frac{\tilde{g}}{t_V d_V}\left(\frac{\partial \delta u}{\partial x} + \frac{\partial \delta v}{\partial y}\right) - \tilde{g}^* \nabla^2 w + \nabla^2 \frac{1}{t_V d_V}\left[\frac{\partial \delta u}{\partial x} + \frac{\partial \delta v}{\partial y}\right] = 0 \tag{12}$$

As the in-plane displacements now appear in both Eqs. (11) and (12) only through the operator divergence$(\boldsymbol{u}_C - \boldsymbol{u}_B)$, the two equations may be combined to obtain a single sixth-order partial differential equation for the transverse deflection of the plate with CLDT.

$$\nabla^6 w - \tilde{g}^*(1 + Y)\nabla^4 w + \frac{(\rho t)_T}{D_1}\nabla^2\left(\frac{\partial^2 w}{\partial t^2}\right) - \tilde{g}^* \frac{(\rho t)_T}{D_1}\left(\frac{\partial^2 w}{\partial t^2}\right) = 0 \tag{13}$$

For equal plate and constraining layer properties, Mead's sandwich equation [7] is recovered. For the case of equal Poisson's ratios, the result is identical to that which Abdulhadi [8] obtained by a different analysis. If the variation of the displacement in one coordinate direction is suppressed and Poisson's ratio is set to zero, the equation collapses to that given by Rao [5] for the beam with constrained layer damping treatment.

III. Method of Equivalent Lengths

A. Simply Supported Plates

We consider first a rectangular plate of side lengths a and b, simply supported on all edges. With the choice of displacement functions for synchronous harmonic oscillation at ω of

$$w = A \sin(p_n x)\sin(q_m y)\cos(\omega t) \tag{14}$$

$$\delta u = u_c - u_b = B\cos(p_n x)\sin(q_m y)\cos(\omega t) \tag{15}$$

$$\delta v = v_c - v_b = C\sin(p_n x)\cos(q_m y)\cos(\omega t) \tag{16}$$

where $p_n = n\pi/a$ and $q_m = m\pi/b$, Eqs. (11) and (12), are both satisfied, as are the conditions of zero moment and transverse displacement at the edges. However, the assumed displacements are appropriate only if the in-plane forces on the ends of the constraining layers are zero. Substitution of Eq. (14) into the sixth-order equation, Eq. (13), shows that the complex valued frequency must satisfy

$$\frac{\omega^2}{\omega_0^2} = \left(1 + \frac{Y g^*}{1 + g^*}\right) \tag{17}$$

The quantity

$$\omega_0^2 = \frac{D_1}{(\rho t)_T}\left(p_n^2 + q_m^2\right)^2 \tag{18}$$

is the frequency of the nth, mth mode of bending vibration of two simply supported rectangular plates having the properties and dimensions of the substrate plate and the constraining layer, but maintained at constant separation without in-plane forces. The parameter Y is as given by Eq. (10) and

$$g^* = \frac{\tilde{g}^*}{p_n^2 + q_m^2} = \frac{G^*}{t_V}\left(\frac{1}{E_C t_C} + \frac{1}{E_S t_S}\right)\frac{1 - \upsilon^2}{\left(p_n^2 + q_m^2\right)} \tag{19}$$

The loss factor of the system is found from

$$\eta_{\mathrm{SYS}} = \frac{\Im\{\omega^2/\omega_0^2\}}{\Re\{\omega^2/\omega_0^2\}} = \frac{Y\Im\{g^*/(1 + g^*)\}}{1 + Y\Re\{g^*/(1 + g^*)\}} \tag{20}$$

Because the value of the coupling parameter, Y, is independent of Poisson's ratio and boundary condition, it is the same for a beam and a plate having the same CLDT. Setting $\sin(q_m y) = 1$ in Eq. (14) and substituting into Eq. (13) with $\nu = 0$ leads to the shear parameter for the simply supported beam of length L as

$$g^*|_{\mathrm{beam}} = \frac{G^*}{t_V (n\pi/L)^2}\left(\frac{1}{E_C t_C} + \frac{1}{E_S t_S}\right) \tag{21}$$

As the lateral dimensions of the plate do not appear in Y, Eq. (20) will yield the same loss factors as for a simply supported beam with the same CLDT if the values of g^* in Eqs. (19) and (21) are the same. This requires that the simply supported beam vibrating in the first mode have L_{EQ} that is related to the dimensions and mode numbers of the plate by

$$\frac{\pi^2}{L_{\mathrm{EQ}}^2} = \frac{1}{1 - \upsilon^2}\left[\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right] \tag{22}$$

The frequency of the plate with CLDT may also be related to the frequency of the beam of equivalent length. After writing Eq. (17) for the plate and for the beam, taking the ratio, and then noting that the values of g^* for the plate and the equivalent beam of the same damping have been made to be equal, the real (observable) frequency of the plate with CLDT is found to be related to that of the equivalent beam with CLDT through

$$\Re\{\omega|_{\mathrm{plate}}\} = \frac{\omega_0|_{\mathrm{plate}}}{\omega_0|_{\mathrm{beam}}}\Re\{\omega|_{\mathrm{beam}}\} \tag{23}$$

Because the frequency of the beam and constraining layer with $g^* = 0$, but maintained as parallel without in-plane force, is

$$\omega_0^2|_{\mathrm{beam}} = \frac{E_S t_S^3 + E_C t_C^3}{12(\rho h)_{\mathrm{EFF}}}\left(\frac{\pi}{L_{\mathrm{EQ}}}\right)^4 \tag{24}$$

and that of the plate and constraining layer similarly uncoupled by the shear layer is

$$\omega_0^2|_{\mathrm{plate}} = \frac{E_S t_S^3 + E_C t_C^3}{12(1 - \upsilon^2)(\rho h)_{\mathrm{EFF}}}\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^2 \tag{25}$$

the ratio of frequencies of the (real) uncoupled systems may be found after substituting Eq. (22) into Eq. (24). Then, upon substitution of the resulting ratio into Eq. (23), we find that the real (observable) frequencies of the plate and the equivalent beam are related through

$$\Re\{\omega|_{\mathrm{plate}}\} = (1 - \upsilon^2)^{1/2}\Re\{\omega|_{\mathrm{beam}}\} \tag{26}$$

Thus, the equivalent simply supported beam with the same CLDT has the same loss factor, but a slightly higher frequency, than does the original simply supported plate.

B. Plates of More General Boundary Conditions

Approximate loss factors for beams with CLDT and any boundary condition are commonly [6] estimated by using the RUK solution for the simply supported beam with the dimensional eigenvalue $n\pi/L$ replaced by the dimensional eigenvalue for the boundary condition of interest. We hypothesize that this practice may be extended to rectangular plates by replacing the squared dimensional eigenvalue of the simply supported plate $(n\pi/a)^2 + (m\pi/b)^2$ with that of the plate having the boundary condition of interest. Using this value to replace the eigenvalue of the simply supported plate in the right-hand side of Eq. (22), we arrive at the equivalent length of a simply supported beam vibrating in the first mode. Using this length with the parameters of the CLDT for the plate of interest, the RUK solution then leads to approximate values for the frequency and loss factor of a rectangular plate with CLDT and any desired boundary condition.

As used here, K_n are the dimensional quantities arising from solving the plate equation, Eq. (13), with $\tilde{g}^*$ set to zero.

$$D_1\nabla^4 w(x,y,t) + (\rho h)_{\mathrm{EFF}}\ddot{w}(x,y,t) = 0 \tag{27}$$

After a separation of variables, $w(x,y,t) = W(x,y)\exp(j\omega t)$,

$$\frac{\nabla^4 W(x,y)}{W(x,y)} = K_n^4 = \frac{(\rho h)_{\mathrm{EFF}}}{D_1}\omega^2 \tag{28}$$

As the nature of the boundary conditions for the plate are such as to preclude obtaining analytical solutions for rectangular plates in all but a few special cases, namely those in which two opposing sides are simply supported [12], explicit expressions for the eigenvalues are obtainable only for these special cases. For other boundary conditions, frequencies are found by numerical means for specified values of aspect ratio and Poisson's ratio. But the eigenvalues may be deduced from such results. For example, in the notation of Leissa [13], dimensionless frequencies are given as values of the quantities $\lambda = \omega a^2\sqrt{\rho/D}$ which, through comparison with Eq. (28), are seen to be related to the required dimensional eigenvalues through

$$K_n^2 = \omega\sqrt{\rho/D} = \lambda/a^2 \tag{29}$$

To summarize the method of equivalent lengths; given a plate with CLDT of known configuration, that is, the moduli, thicknesses, and Poisson's ratio of the substrate and constraining layer, and the complex modulus and thickness of the shear layer, we first compute the parameters Y and $\tilde{g}^*$ as given by Eqs. (9) and (10). Then, for the desired boundary condition and mode, we determine the dimensional eigenvalue for the plate mode and geometry of interest and substitute it for the dimensional eigenvalue of the simply supported plate in the right factor of Eq. (22) to find the length of the equivalent simply supported beam from

$$\frac{\pi^2}{L_{\mathrm{EQ}}^2} = \frac{K_n^2}{1-\upsilon^2} \tag{30}$$

We then use this to evaluate the loss factor from Eq. (20), with

$$g^* = \tilde{g}^*(L_{\mathrm{EQ}}/\pi)^2 = \tilde{g}^*(1-\upsilon^2)/K_n^2 \tag{31}$$

The frequency of the plate is found from Eq. (26) after finding the frequency of the equivalent beam. From Eqs. (17) and (24), this is

$$\Re\{\omega_{\mathrm{beam}}\}_{\mathrm{EQ}} = \omega_0|_{\mathrm{beam}}\Re\left\{\sqrt{1+\frac{Yg^*}{1+g^*}}\right\}$$
$$= \sqrt{\frac{E_S t_S^3 + E_C t_C^3}{12(\rho h)_{\mathrm{EFF}}}}\left(\frac{K_n^2}{1-\upsilon^2}\right)\Re\left\{\sqrt{1+\frac{Yg^*}{1+g^*}}\right\} \tag{32}$$

Note that this frequency differs slightly from that obtained the common practice of taking the real frequency to be the square root of the real part of the squared complex frequency, rather than the real part of the square root.

IV. Validation

To evaluate the effectiveness of the method of equivalent lengths in estimating the loss factors for plates with constrained layer damping treatments and arbitrary boundary conditions, a finite element representation of the plate with CLDT was used. Loss factors and frequencies were determined from the complex eigenvalues as determined with Nastran 2004, using three-dimensional, eight-noded solid brick elements (CHEXA) with each node having three transnational degrees of freedom.

Material parameters used were as follows: For the substrate plate and constraining layer, Young's modulus (E_C, E_S) was taken as 10 Mpsi (69 GPa), Poisson's ratio (υ) as 0.33, and the weight density (ρ_C, ρ_B) to be 0.1 lb/in^3 (2730 kg/m^3). The thickness of the plate (t_S) was taken as 0.060 in. (1.524 mm); and that of the constraining layer (t_C) as 0.005 in. (0.127 mm). The shear (damping) layer was taken (t_V) as 0.005 in. (0.127 mm) thick, with loss factor (η_{MAT}) of 1.0 and a weight density (ρ_V) of 0.035 lb/in.^{3} (956 Kg/m^3). The real part of the complex shear modulus (G) was taken as 100 or 300 psi (0.69 or 2.07 MPa), as necessary to produce maximum damping within the range of modes considered. The same value was used at all frequencies. The length, a, of the plate was taken as 7 in. (178 mm), and various aspect ratios $(AR = a/b)$ between 0.5 and 3.5 were considered. The CLDT was taken as fully covering the plate on one side only.

As the plate length was held fixed, and square elements (side length $\frac{1}{6}$ in. or 4.23 mm) were used in all cases, the number of elements varied from 1512 at $AR = 3.5$ to 10,584 at $AR = 0.5$. As one element was taken through the thickness of each layer, the element aspect ratio in the shear and constraining layers was 33.3, and less than 3 in the substrate. In the case of a free or simply supported edge, both components of the in-plane displacement were allowed to develop freely. In the case of a clamped edge, both components were fully restrained. For a clamped or simply supported edge, the transverse displacement of the base plate was restrained. Further details of the finite element model (and its validation) are given elsewhere [14].

Loss factors were obtained from the complex eigenvalues found through a finite element analysis, rather than by using the method of modal strain energy, as it is well established [14] that the latter method leads to significant overpredictions of system loss factors when shear layer materials have loss factors in the range of greatest interest, that is, $\eta_{\mathrm{MAT}} \cong 1$. It was also demonstrated, however, that loss factors and frequencies for the first six modes of the cantilever beam, when extracted from complex frequencies found by a finite element analysis, are in satisfactory agreement (within 3%) with Rao's exact solution. FEA was found to lead small underestimates of loss factors and small overpredictions of frequencies.

For each of six combinations of plate boundary conditions, plates of several aspect ratios were considered. Each combination of boundary condition and aspect ratio results in a multiplicity of modes with different frequencies and nodal patterns. Frequencies and loss factors were evaluated for all such modes having frequencies less than (about) 3 times the frequency of maximum damping for that particular combination of boundary condition and aspect ratio.

A. Simply Supported Plates

As the plate simply supported on all four sides (SSSS) was seen to collapse to the RUK solution for the simply supported beam, loss factors and frequencies obtained by finite element analysis (FEA) should coincide with results from the RUK analysis. Such a comparison is shown in Fig. 2. Results coded CE (complex eigenvalues) were obtained by Nastran, with the loss factor computed from the complex eigenvalues. Results coded MEL were obtained by the RUK solution, with the equivalent length for each mode and aspect ratio determined from Eq. (22). Results from the MEL solution fall on the solid line for all modes and all values of aspect ratio.

In general, the loss factors from finite elements are typically somewhat lower than those from the MEL solution. Although the agreement is quite satisfactory for the first several modes of the

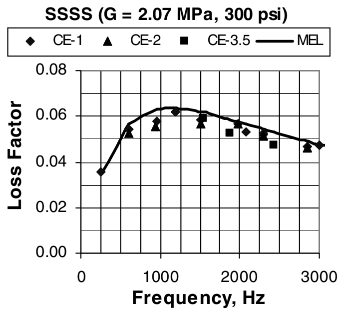

Fig. 2 Loss factors for simply supported plates as predicted by FEA and MEL solutions.

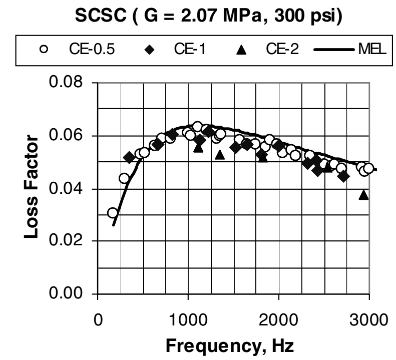

Fig. 3 Loss factors for simply supported-clamped plates (SCSC) as predicted by FEA and MEL solutions.

square plate, more significant differences are seen for higher modes. In the case of the square plate, the agreement between the numerical values and those from the extension to the RUK solution is found to be quite good for the modes having nodal lines in only one direction, and not as good for others. Agreement for the square plate CE-1 is somewhat better than in the results for the higher aspect ratios, coded CE-2 and CE-3.5. In general, agreements between the two predictions were seen to diminish with increasing complexity of mode shape.

Values of the loss factor for the first six flexural modes of a simply supported square plate with thick core and thin face sheet have been given [15] by Cupiał and Nizioł and compared with results obtained [16] by Rao and Nakra. Application of the MEL to the same configuration was found to lead to loss factors differing by no more than one unit in the third significant digit from these previously obtained results. Cupiał and Nizioł have also given [15] a comparison of loss factors and frequencies for five modes of a simply supported symmetric sandwich plate with thin core. Loss factors obtained by the MEL were found to be identical; frequencies found by the MEL, using Eqs. (26) and (32), were found to be higher by about $\frac{1}{3}$ and $\frac{1}{2}$%, respectively, than the previous results.

As the MEL is based on an adaptation of the RUK solution, in the case of the simply supported plate it should represent the exact solution to within the assumptions of the RUK theory. Thus, discrepancies must either be attributed to an inability of FEA to effectively model the CLDT, or to inadequacies of the RUK solution. To verify again that the discretization was adequate, the calculation for $a/b = 2$ was repeated with the number of elements doubled in both directions. The use of smaller elements also reduced the aspect ratio of the elements modeling the thin shear layer from 33 to 16.5. The loss factors for the first six modes evaluated (the triangular data points in Fig. 2) were found to change by less than 0.5%. That the FEA predictions are lower is believed to be a consequence of the simplified RUK displacements. Loss factors for the cantilever beam obtained by FEA have been found [17] to be lower than those computed from Rao's exact solution [5], which is exact to within the classic assumptions.

This comparison suggests that the differences between the MEL solution and the results obtained from FEA are due to the inability of the simplified displacement fields used in the classic RUK solution to adequately describe the displacement fields occurring in the higher modes of plate vibration, even in the case of simply supported edges. As better agreements should not be expected from applying the MEL to obtain approximate solutions for other boundary conditions, it would appear that agreement between results from the finite element analysis and from the method of equivalent lengths should not be expected to be any better than about 15%, and that overpredictions of loss factors may generally be expected.

B. Simply Supported Ends, Clamped Sides

Perhaps the simplest example (other than SSSS) of a rectangular plate having no free edges is that for which the ends ($x = 0$ and $x = a$) are simply supported and the sides ($y = 0$ and $y = b$) are clamped (SCSC). As the solution is separable, Voigt's method [12] may be applied. The transcendental equation for the eigenvalues as given by Leissa [13] was developed and solved numerically for three values of aspect ratio: $a/b = \frac{1}{2}$, $a/b = 1$, and $a/b = 2$. In this case, the eigenvalues are independent of Poisson's ratio. Equivalent lengths, as determined with these eigenvalues and Eq. (30), were used in the RUK solution to obtain loss factors and frequencies. The results are compared in Fig. 3 with the results from CE as found by FEA for plates of several aspect ratios.

The agreement at an aspect ratio of $\frac{1}{2}$ is particularly good, but the agreement decreases with increasing aspect ratio, with discrepancies approaching 30% for an aspect ratio of 2. In view of the fact that the end condition for the constraining layer in the RUK solution is not a true match of the clamped end, agreement at $a/b = 1$ is surprisingly good. The much poorer agreement at $a/b = 2$ may be a consequence of the increased influence of the clamped edges on the transverse displacements. The first mode with longitudinal nodal line appears for $a/b = 2$ at just below 3000 Hz. The increased complexity of such modes appears to increase further the discrepancy between the results of the two calculations.

C. Simply Supported Plate with Free End

The solution for the plate with the end ($x = 0$) and sides ($y = 0$ and $y = b$) simply supported, but with a free edge ($x = a$) is also separable (SSSF), and Voigt's method may again be used to obtain the transcendental equation for the eigenvalues. Because of the inclusion of a free edge, the boundary conditions (and the eigenvalues) are dependent on the value of Poisson's ratio. The transcendental equation as given by Leissa [13] was developed and solved numerically for the eigenvalues. Over 150 were computed for $\upsilon = 0.33$ and several aspect ratios. These were used to determine loss factors and frequencies from the RUK solution by using the equivalent length, Eq. (30), for each mode and aspect ratio. The results are compared in Fig. 4 with the findings from finite element analysis for plates of several aspect ratios.

In this case, the agreement is quite good for plates of low aspect ratio (i.e., $a/b \leq 1$), even in the higher modes, but decreases notably at the higher aspect ratios (i.e., $a/b \geq 2$). In this frequency range,

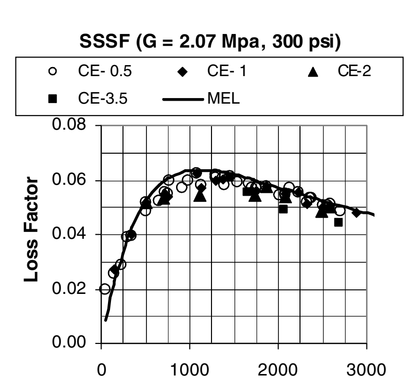

Fig. 4 Loss factors for simply supported plates with free edge as predicted by FEA and MEL solutions.

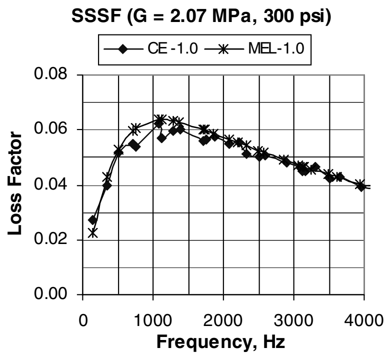

Fig. 5 Loss factors for square plates with one free edge (SSSF).

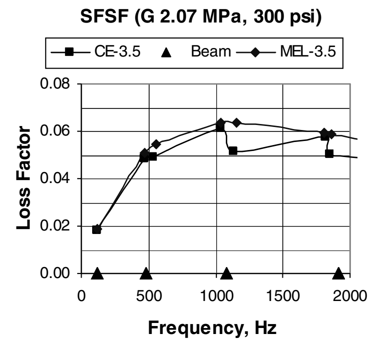

Fig. 7 Loss factors for SFSF plates, $a/b = 3.5$.

modes for $a/b = 3.5$ include only those with no longitudinal nodes, that is, $m = 1$ only, whereas mode shapes at $a/b = 2$ are more complex.

The predictions of loss factor and frequency for $a/b = 1$ are shown in greater detail in Fig. 5. The agreement for some modes is very good, that is, the sixth ($n = 1$, $m = 3$), but is significantly less so for the seventh ($n = 3$, $m = 2$).

D. Simply Supported Ends, Free Edges

As the plate with simply supported ends at $x = 0$ and $x = a$ and free sides at $y = 0$ and $y = b$ (SFSF) also has a separable solution, the transcendental equation for the eigenvalues may be obtained. Once again, however, the eigenvalues depend on Poisson's ratio. The transcendental equation as given elsewhere [13] was developed and solved numerically for the eigenvalues at $\upsilon = 0.33$. These were used to determine the equivalent length for each mode and aspect ratio. Loss factors for plates of several aspect ratios as found with the MEL solution are compared in Fig. 6 with the findings from finite element analysis.

Once again, the agreement in the case of the plate with low aspect ratio ($a/b = \frac{1}{2}$) is quite good, with deviations becoming more significant for higher modes and higher aspect ratios. Significant discrepancies occur for $a/b = 2$ when the longitudinal nodal line appears.

In other work [18], the same finite element model was applied to a evaluation of loss factors and frequencies for the beamlike bending modes of narrow plates ($a/b = 14$) with free sides and simply supported ends. In that case, the CLDT was the same as that of the present, except that a value of $G = 100$ psi (0.69 MPa) was used. Discrepancies between the MEL and finite element solutions were found to increase with mode number, but remained below 3% for all of the first five modes of this type ($\Re\{\omega\} < 2700$ Hz). It is therefore concluded that the simplifying assumptions of the RUK theory lead to less satisfactory approximations of the deformation fields at higher modes and with the more complex transverse variations in displacement occurring in wider plates.

An examination of the results at $a/b = 3.5$ is of particular interest. The first four frequencies of a simply supported beam of the same length, but without CLDT, are shown for reference on the abscissa of Fig. 7. Two modes of the damped plate occur near each of the second, third, and fourth beam frequencies. For the symmetric mode without longitudinal nodes, the solutions (CE) from finite element analysis are in quite good agreement with the values obtained by using MEL. However, the agreement in both loss factor and frequency is not as good for the adjacent mode with a more complex displacement field.

E. Cantilever Plate

The cantilever plate, that is, the plate with one clamped and three free edges (CFFF or CANT), is of particular interest. It is often used in experimental investigations as the boundary condition can be quite simply simulated in the laboratory. Results for three values of aspect ratio are given in Fig. 8. Note that the finite element results (CE) are for $\upsilon = 0.33$, whereas the MEL results found with an equivalent length are for $\upsilon = 0.30$.

In this case, however, there is no means of developing an analytical solution from which the eigenvalues may be determined. Accordingly, approximate eigenvalues for a plate with Poisson's ratio of 0.3 were taken from the literature [13] and used to determine the equivalent lengths for various modes at several aspect ratios. The real part of the modulus of the shear (damping) layer was reduced to $G = 100$ psi (0.69 MPa) for these calculations to move the damping peak to lower modes.

In general, the agreements at the lowest modes are about the same as for the other boundary conditions. However, for the modes near the damping peak the discrepancy appears to exceed 20%. The mode with anomalously high damping (CE-2 at $\approx$1100 Hz) is a two-stripe or chord wise bending mode. This mode did not show exceptionally high damping at lower aspect ratios.

Once again, it is of interest to examine in some greater detail the results at a higher aspect ratio. The finite element results (CE) are shown in Fig. 9 for an aspect ratio of 3.5. The frequencies of the first three modes of the bare beam ($a/b = \infty$) are also shown, and it is seen that the plate with CLDT has two modes near each of the second and third beam frequencies. For one if these, a beamlike bending mode, the agreement in loss factor is quite good, whereas for the other, a more complex mode shape with nodal line, the discrepancy is much greater. A noticeable difference in frequency also appears at

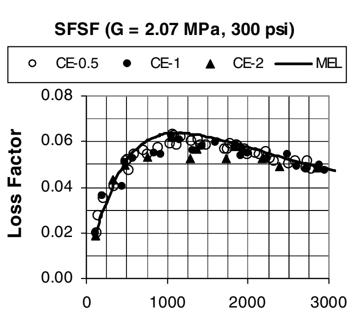

Fig. 6 Loss factors for simply supported free plates as predicted by FEA and MEL solutions.

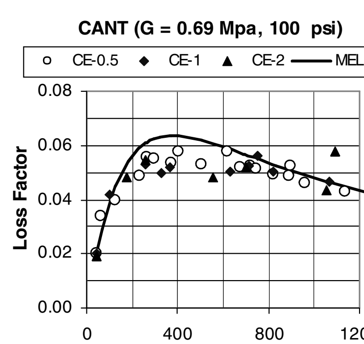

Fig. 8 Loss factors for cantilever plates as predicted by FEA and MEL solutions.

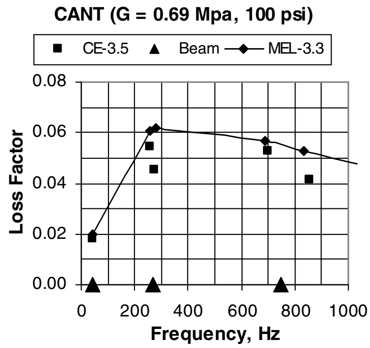

Fig. 9 Loss factors for CFFF plates, $a/b \approx 3.5$.

the highest modes shown. As in the case of the SCSC plate, the end condition for the constraining layer at the clamped edge does not match that which is implicit in the RUK analysis for the simply supported beam.

F. Free–Free Plate

The final boundary condition considered is that of the rectangular plate, free on all four sides (FFFF). No analytical solution from which eigenvalues may be extracted is available. However, as in the case of the cantilever plate, extensive numerical studies have been undertaken for the determination of approximate frequencies. Such values have been assembled by Leissa [13] from various sources. Loss factors and frequencies determined from the complex frequencies obtained by finite element analysis are shown in Fig. 10, and are compared with the predictions of the RUK analysis using equivalent lengths as determined from Eq. (30) with the approximate eigenvalues. Results from finite elements (CE) are for $\upsilon = 0.33$; results from MEL are for $\upsilon = 0.30$.

For this boundary condition, significant differences are seen at the lower frequencies. The anomalous behavior of the first two modes, wherein the first mode of the square plate (CE-1) has higher damping than the second, is a reversal of general trend and is without explanation. For the higher modes, agreement between the two solutions is similar to that observed for other boundary conditions.

V. Application to Design

For the purpose of evaluating the accuracy of predictions from the MEL, system parameters were deliberately chosen so that the damping peak occurred at relatively high modes. In an actual design process it is likely that certain modes are of particular interest, but the initial choice of parameters would lead to a damping peak at other than the desired modes (frequencies). However, an estimate of the necessary parameters can be obtained from a preliminary calculation. Because the location of the maximum is governed by the value of the shear parameter, g^* in Eq. (19), that is inversely proportional to the square of the eigenvalue, and as the frequency is proportional to the squared eigenvalue, the damping peak may be moved to a different mode by selecting a new combination of the parameters in the real part of $\tilde{g}^*$; see Eq. (9). The choice of a new value for the ratio of G/t_V (taking into account the actual frequency dependence of the storage modulus), changed in proportion to the ratio of desired and present frequencies of the maximum loss factor, will then move the maximum loss factor close to the desired mode. This process may be used to reduce significantly the number of iterations required in the actual calculations upon which the final design will be based.

As an example, we consider the predictions for the SCSC and FFFF modes. In both cases, the initial choice of parameters (including $G = 300$ psi or 2.07 MPa) led to a damping peak at the modes having frequencies around 1200 Hz. See Figs. 3 and 10. The shear parameter, Eq. (21), is proportional to the square of the length, or to the inverse of frequency. If it is desired to move the damping peak to the modes having frequencies of about 400 Hz, the longer wavelength of the lower modes will lead to a shear parameter that is too high for optimal damping. Thus, the shear parameter must be reduced by the same factor as the desired frequency change. This may be achieved by increasing the thickness of the shear layer by a factor of 3 or by reducing the shear modulus by a factor of 3. The loss factors and frequencies found by the method of effective lengths and FEA for the SCSC and FFFF plates with the same CLDT as used to obtain the results of Figs. 3 and 10, except for the reduction in the storage modulus of the shear layer to $G = 100$ psi (0.69 MPa), are shown in Figs. 11a and 11b.

As may be seen from Fig. 11a, this process is quite effective in moving the damping peak to the desired modes in the case of the simply supported, clamped plate. In the case of the FFFF plate of Fig. 11b, the adjustment in shear parameter led to a doubling of the loss factors of the lowest modes. However, the frequency of the modes at the damping peak has been lowered to about $\frac{1}{2}$, rather than the desired $\frac{1}{3}$.

A comparison of the success of the MEL in predicting the peak loss factors for different boundary conditions (SCSC, Fig. 3; SSSF, Fig. 4; SFSF, Fig. 6; CFFF, Fig. 8; FFFF, Fig. 10) suggests that the deviations from the FEA predictions generally increase as the number of free edges is increased. This trend is also evident in Figs. 11a and 11b. Jones [6] noted, in the case of beams, that the effectiveness of the predictions for the lower modes could be improved if empirical correction factors were applied in the process of eigenvalue replacement. The comparisons given here suggest that

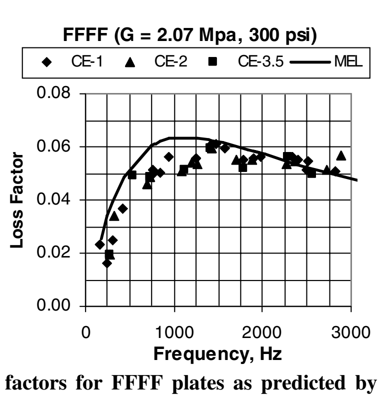

Fig. 10 Loss factors for FFFF plates as predicted by finite element analysis and MEL solutions.

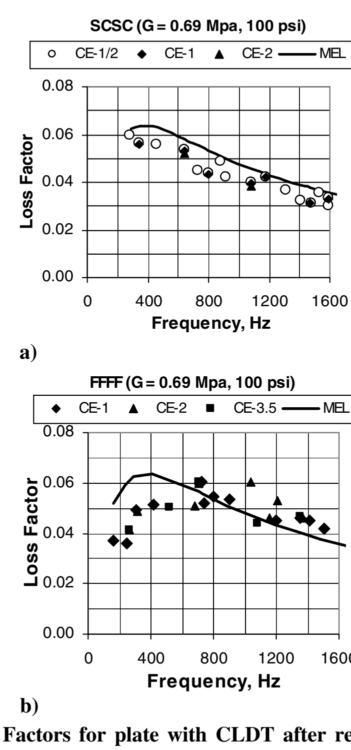

Fig. 11 Loss Factors for plate with CLDT after reducing the shear modulus: a) opposing sides clamped (SCSC); b) all edges free (FFFF).

such a process might also be useful in the case of plates, but more so in the presence of free, rather than clamped, edges.

VI. Limitations

The comparisons given here of loss factors and frequencies extracted from a finite element analysis with those resulting from the application of the method of equivalent lengths are subject to several limitations.

1) Only a single configuration for the constrained layer damping treatment was considered. However, the loss factor for a CLDT is governed by just two parameters, the shear parameter g^* and a coupling parameter Y. As the first of these contains the eigenvalue, it serves as a surrogate variable for wavelength. By considering many modes, that is, many wavelengths, significant variations in the shear parameter g^* have in fact been considered. But the single value of the coupling parameter ($Y = 0.360$) used in all these comparisons is that characteristic only of the constrained layer damping treatments applied on existing structures to increase damping. In the case of the symmetric sandwich beam with thin core, for example, the value of Y is typically greater than three. Thus, the observations made here may not be applicable to such configurations.

2) The complex modulus of the shear layer was taken to be the same at all frequencies. Thus, the calculations given here do not represent the actual behavior of a CLDT with a truly viscoelastic shear layer for which the storage modulus and loss factor would vary significantly with frequency. The results are applicable only for the purpose intended, that is, the mode by mode (i.e., frequency by frequency) comparison of predictions by two methods.

3) In the finite element analysis for each plate boundary condition, only one end condition for the constraining layer was considered. For a simply supported or free edge, the constraining layer end was taken to be free of in-plane force so that the strain in the shear layer at the end is not zero. For the clamped edge, the end of the constraining layer was taken as being fixed to the plate so that the shear strain at that end is zero. The method of equivalent lengths cannot account for other constraining layer end conditions, and these influences can be significant [18,19].

4) Finally, only the classic RUK model was employed. Although it is likely that improved solutions, taking into account such factors as the inability of the soft shear layer to maintain constant separation between constraining layer and plate [20], would improve agreement; the increased complexity would compromise the ability to obtain the quick estimates necessary for use in preliminary design.

VII. Conclusions

A method of equivalent lengths has been considered and evaluated as a means for providing quick estimates of the loss factors and frequencies of rectangular plates with general boundary conditions, fully covered on one side with a CLDT.

The motivation for this approach arises from two observations. First, that the expressions for the loss factor and frequency of the simply supported plate with CLDT and the simply supported beam with CLDT take identical forms. The second observation is that, in both cases, the complex frequency (from which the natural frequency and the loss factor may be found) depends on only three parameters: 1) a coupling coefficient, dependent only on a certain combination of the parameters of the CLDT and not on the lateral dimensions; 2) the dimensional eigenvalue, that does depend on the lateral dimensions, but not on the CLDT parameters; and 3) a shear parameter, which depends both upon a second combination of the parameters of the CLDT and the lateral dimension through the eigenvalue.

The second observation was used to suggest that the loss factor and frequency of a rectangular plate of any boundary condition and CLDT might be approximated by those of an equivalent simply supported plate with the same CLDT and eigenvalue. The first observation was then used to suggest that that equivalent simply supported plate might be related to a simply supported beam of an equivalent length, the loss factor and frequency of which might be found from the results of the Ross, Ungar, and Kerwin analysis. Thus, through this two-step process, an estimate of the loss factor and frequency of a rectangular plate with constrained layer damping treatment and any boundary conditions may be found from the RUK solution.

Although the relationship between loss factor and frequency resulting from the method of equivalent lengths is the same for all aspect ratios and modes for a given CLDT configuration, it is still necessary to know either the eigenvalue a priori for the aspect ratio, boundary condition, and mode of interest, or to determine it from the bare plate frequency of the plate to which the damping treatment is to be applied. This is necessary so that the eigenvalue of the specific mode of interest may be used to determine the equivalent length for use in the RUK solution for the loss factor and frequency.

Predictions obtained through the method of equivalent lengths were compared with evaluations by finite element analysis of frequencies and loss factors of rectangular plates with various boundary conditions. In addition to the simply supported plates, consideration was given to the plate with two edges simply supported and two edges clamped, three edges simply supported and one edge free, two edges simply supported and two edges free, one edge clamped and three edges free, and to the plate with four free edges. In general, the estimates tended to be somewhat less satisfactory when the number of free edges was increased. With the exception of the lowest modes of the free–free plate, agreement over a wide range of modes and aspect ratios was within 20%, with isolated instances of differences up to about 30%. Some portion of the differences are believed to be an inability of the simplified displacements assumed in the RUK analysis to adequately represent the more complex mode shapes arising in plates, especially in the cases of the higher modes and larger aspect ratios.

However, the method of effective lengths has been shown [14] to predict system loss factors for plates with no free edges that are in much better agreement with finite element results using complex eigenvalues than does the method of modal strain energy [21]. But for cantilever plates and plates with all edges free, the new method was found to be only slightly superior.

Nonetheless, it appears that estimates of loss factors and frequencies, suitable for the preliminary design of rectangular plates with constrained layer damping treatments, may be obtained through the use of the equivalent length and the RUK solution. In general, the approximation is somewhat less satisfactory in the case of the cantilever plate, and significantly less so in the case of the plate free on all edges. Caution should also be exercised if the ends of the constraining layer are not free, as is tacitly assumed in the RUK solution. However, it is to be expected that the details of the end condition should become less significant for the higher modes, as the influence of the end condition should diminish with an increasing number of half wavelengths in the displacement field.

In addition to providing a useful approximation (although generally an overestimate) of the loss factor, the method of equivalent lengths also appears useful in identifying the configuration at which the maximum loss factor is achieved. With the exception of the free–free plate, the frequency at which the MEL predicted a maximum in the loss factor was in quite good agreement with the frequency (i.e., mode) for which the finite element analysis predicted maximum damping.

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K. Shivakumar
Associate Editor